

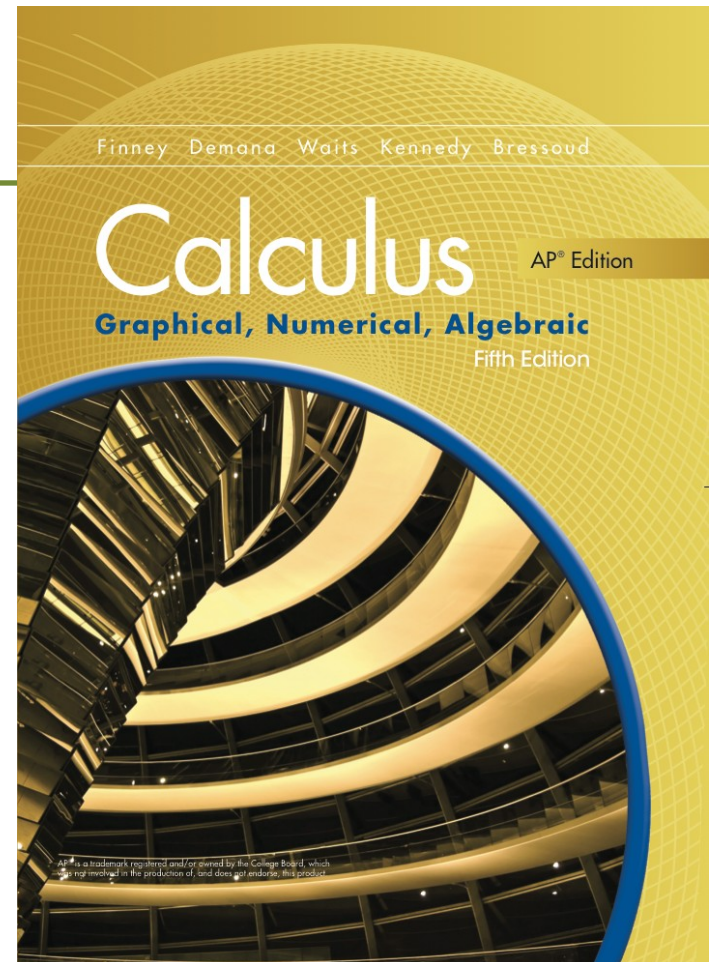
# Chapter 11

## Parametric, Vector, and Polar Functions



### Section 11.3

### Polar Functions



# Quick Review

1. Find the component form of a vector with magnitude 4 and direction angle  $30^\circ$ .
2. Find the area of a  $30^\circ$  sector of a circle of radius 6.
3. Find the area of a sector if a circle of radius 8 that has a central angle of  $\pi/8$  radians.
4. Find the rectangular equation of a circle of radius 5 centered at the origin.

# Quick Review

Given  $x = 3\cos t$ ,  $y = 5\sin t$ ,  $0 \leq t \leq 2\pi$ .

5. Find  $dy/dx$
6. Find the slope of the curve at  $t = 2$ .
7. Find the points on the curve where the slope is zero.
8. Find the points on the curve where the slope is undefined.
9. Find the length of the curve from  $t = 0$  to  $t = \pi$ .

# Quick Review Solution

1. Find the component form of a vector with magnitude 4 and direction angle  $30^\circ$ .  $\langle 2\sqrt{3}, 2 \rangle$
2. Find the area of a  $30^\circ$  sector of a circle of radius 6.  $3\pi$
3. Find the area of a sector if a circle of radius 8 that has a central angle of  $\pi/8$  radians.  $4\pi$
4. Find the rectangular equation of a circle of radius 5 centered at the origin.  $x^2 + y^2 = 25$

# Quick Review Solutions

Given  $x = 3\cos t$ ,  $y = 5\sin t$ ,  $0 \leq t \leq 2\pi$ .

5. Find  $dy/dx$  -  $5/3 \cot t$

6. Find the slope of the curve at  $t = 2$ . -  $5/3 \cot 2$

7. Find the points on the curve where the slope is zero.  $(0, 5)$  and  $(0, -5)$

8. Find the points on the curve where the slope is undefined.  $(3, 0)$  and  $(-3, 0)$

9. Find the length of the curve from  $t = 0$  to  $t = \pi$ .  $\approx 12.763$

# What you'll learn about

- The slope of a curve given by a polar function
- Areas bounded by polar curves

## ...and why

Polar equations enable us to define some interesting and important curves that would be difficult or impossible to define in the form  $y=f(x)$ .

# Example Rectangular and Polar Coordinates

Find rectangular coordinates for the point with the polar coordinate  $(4, \pi/2)$ .

$(0,4)$

# Example Rectangular and Polar Coordinates

Find two different sets of polar coordinates for the point with the rectangular coordinate  $(-3, 3)$ .

A point has infinitely many sets of polar coordinates, here are two typical examples:  $(3\sqrt{2}, 3\pi/4)$  and  $(-3\sqrt{2}, -\pi/4)$



# Polar-Rectangular Conversion Formulas

$$x = r \cos \theta \qquad r^2 = x^2 + y^2$$

$$y = r \sin \theta \qquad \tan \theta = \frac{y}{x}$$

# Parametric Equations of Polar Curves

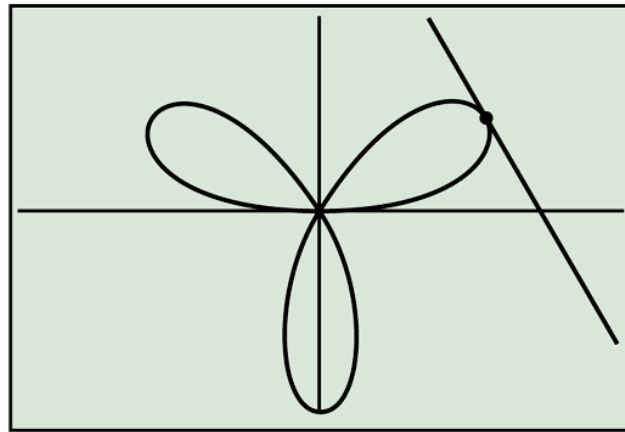
The polar graph of  $r = f(\theta)$  is the curve defined parametrically by:

$$x = r \cos \theta = f(\theta) \cos \theta$$

$$y = r \sin \theta = f(\theta) \sin \theta$$

# Example Finding Slope of a Polar Curve

Find the slope of the rose curve  $r = 2\sin 3\theta$  at the point where  $\theta = \pi / 6$ .



$[-3, 3]$  by  $[-2, 2]$

$0 \leq \theta \leq \pi$

$$\left. \frac{dy}{dx} \right|_{\theta = \frac{\pi}{6}} = \left. \frac{dy/d\theta}{dx/d\theta} \right|_{\theta = \frac{\pi}{6}} = \frac{\left. \frac{d}{d\theta} (2\sin 3\theta \sin \theta) \right|_{\theta = \frac{\pi}{6}}}{\left. \frac{d}{d\theta} (2\sin 3\theta \cos \theta) \right|_{\theta = \frac{\pi}{6}}} = -\sqrt{3}$$

# Area in Polar Coordinates

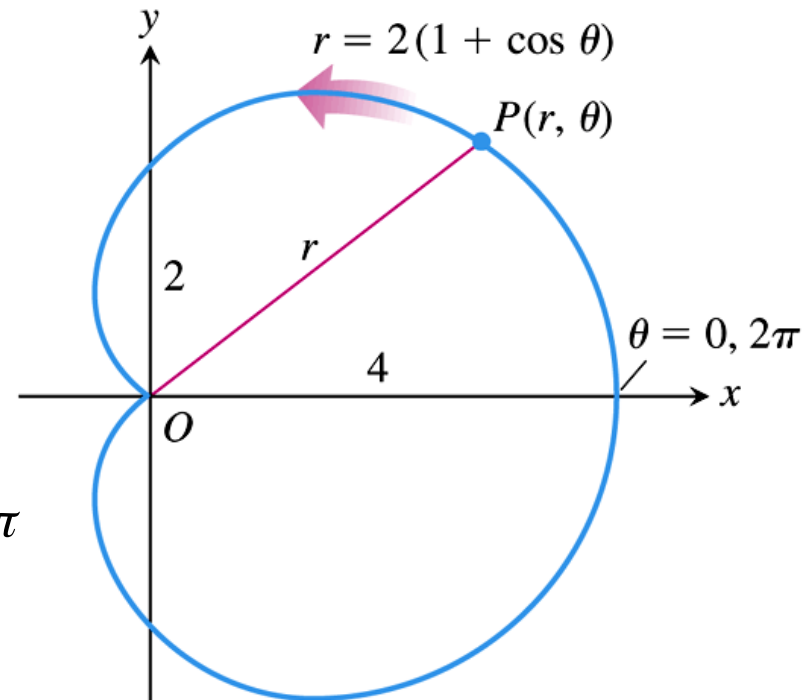
The area of the region between the origin and the curve  $r = f(\theta)$  for  $\alpha \leq \theta \leq \beta$  is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 d\theta.$$

# Example Finding Area

Find the area of the region in the plane enclosed by the cardioid  
 $r = 2(1 + \cos \theta)$ .

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos \theta)^2 d\theta \\ &= \int_0^{2\pi} (3 + 4\cos \theta + \cos 2\theta) d\theta \\ &= \left[ 3\theta + 4\sin \theta + \frac{\sin 2\theta}{2} \right]_0^{2\pi} = 6\pi - 0 = 6\pi \end{aligned}$$

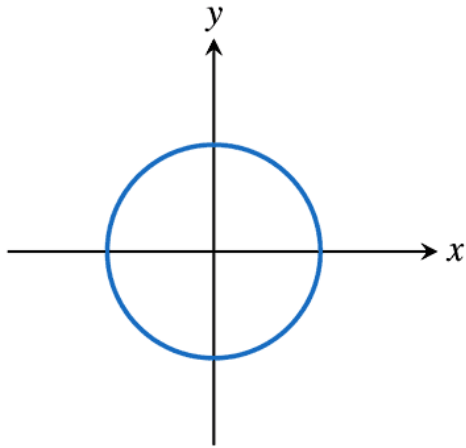


# Area Between Polar Curves

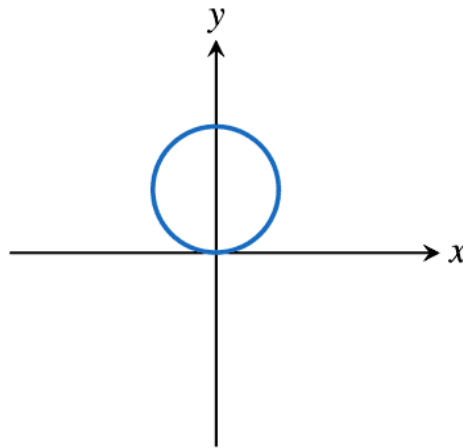
The area of the region between  $r_1(\theta)$  and  $r_2(\theta)$  for  $\alpha \leq \theta \leq \beta$  is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} (r_2(\theta))^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} (r_1(\theta))^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta.$$

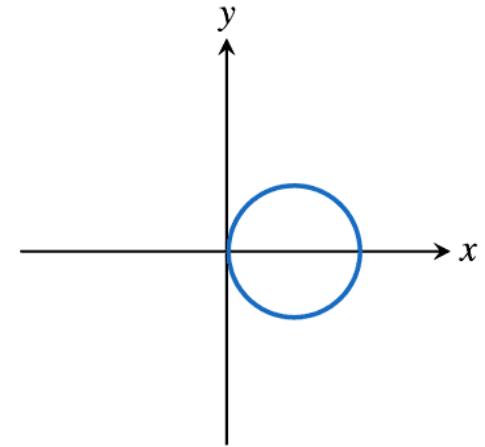
# Circles



$$r = \text{constant}$$
$$0 \leq \theta \leq 2\pi$$

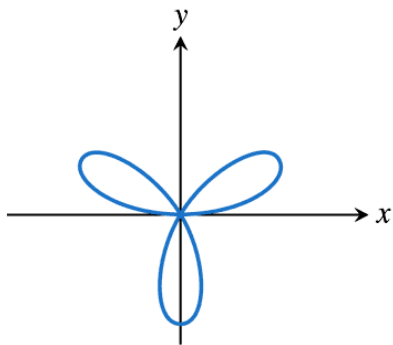


$$r = a \sin \theta$$
$$0 \leq \theta \leq \pi$$

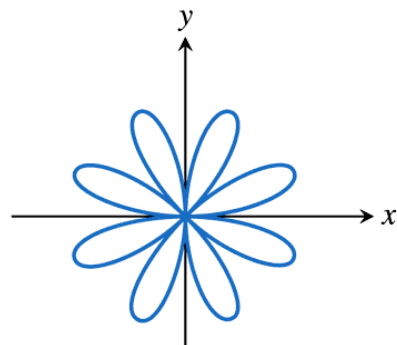


$$r = a \cos \theta$$
$$0 \leq \theta \leq \pi$$

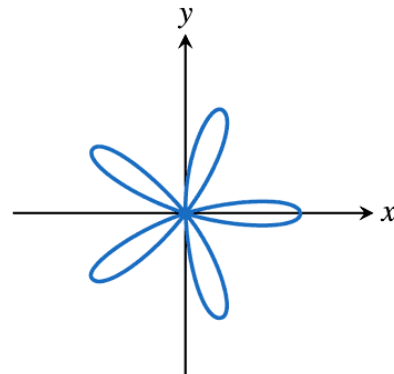
# Rose Curves



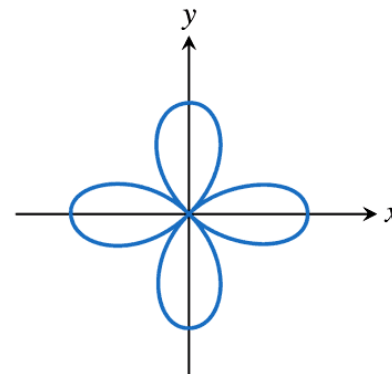
$r = a \sin n\theta, n \text{ odd}$   
 $0 \leq \theta \leq \pi$   
 $n$  petals  
 y-axis symmetry



$r = a \sin n\theta, n \text{ even}$   
 $0 \leq \theta \leq 2\pi$   
 $2n$  petals  
 y-axis symmetry and  
 x-axis symmetry



$r = a \cos n\theta, n \text{ odd}$   
 $0 \leq \theta \leq \pi$   
 $n$  petals  
 x-axis symmetry



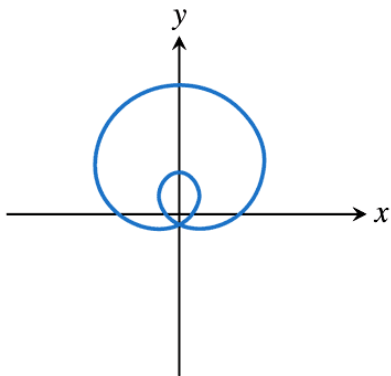
$r = a \cos n\theta, n \text{ even}$   
 $0 \leq \theta \leq 2\pi$   
 $2n$  petals  
 y-axis symmetry and  
 x-axis symmetry



# Limaçon Curves

$r = a \pm b \sin \theta$  or  $r = a \pm b \cos \theta$  with  $a > 0$  and  $b > 0$

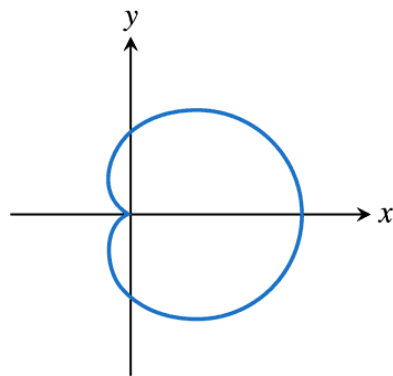
( $r = a \pm b \sin \theta$  has y-axis symmetry;  $r = a \pm b \cos \theta$  has x-axis symmetry.)



$$\frac{a}{b} < 1$$

$$0 \leq \theta \leq 2\pi$$

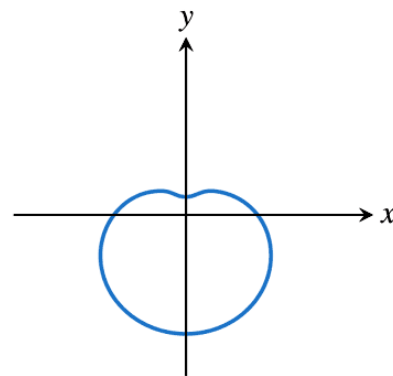
Limaçon with loop



$$\frac{a}{b} = 1$$

$$0 \leq \theta \leq 2\pi$$

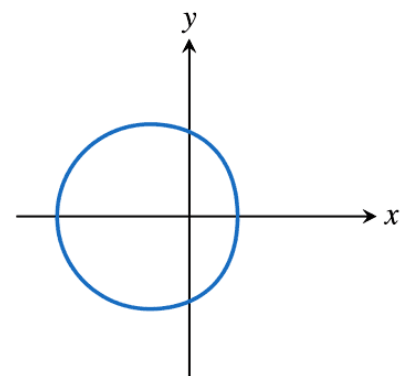
Cardioid



$$1 < \frac{a}{b} < 2$$

$$0 \leq \theta \leq \pi$$

Dimpled limaçon

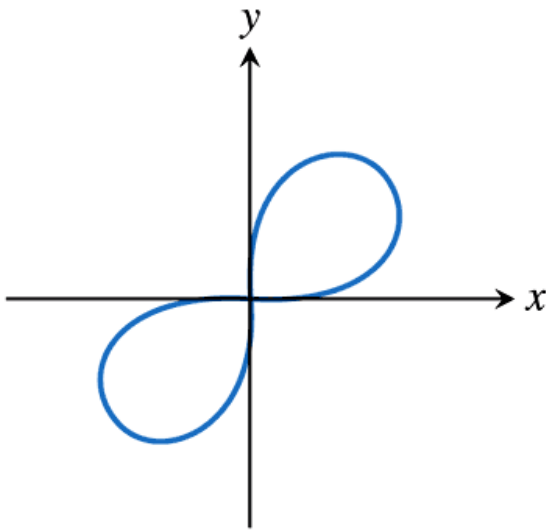


$$\frac{a}{b} \geq 2$$

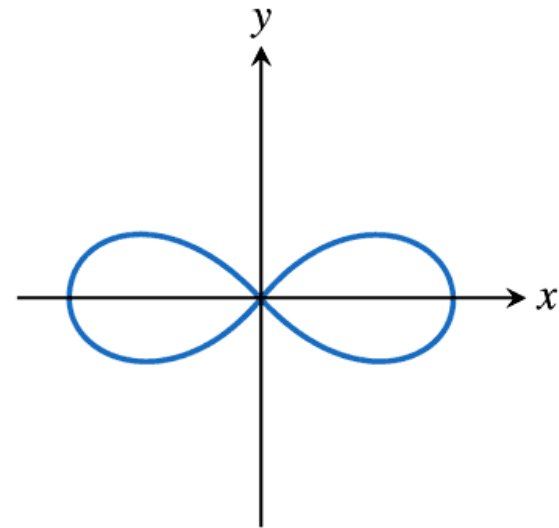
$$0 \leq \theta \leq 2\pi$$

Convex limaçon

# Lemniscate Curves

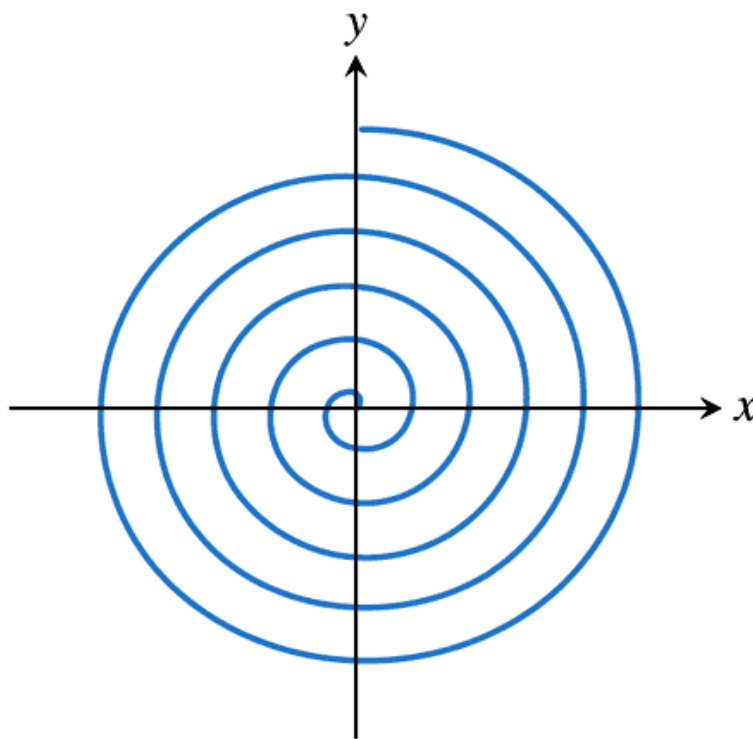


$$r^2 = a^2 \sin 2\theta$$
$$0 \leq \theta \leq \pi$$



$$r^2 = a^2 \cos 2\theta$$
$$0 \leq \theta \leq \pi$$

# Spiral of Archimedes



$$r = \theta > 0$$

# Quick Quiz Sections 11.1 – 11.3

You may use a graphing calculator to solve the following problems.

1. Which of the following is equal to the area of the region inside the polar curve  $r = 2\cos\theta$  and outside the polar curve  $r = \cos\theta$ ?

(A)  $3 \int_0^{\pi/2} \cos^2 \theta \, d\theta$

(B)  $3 \int_0^{\pi} \cos^2 \theta \, d\theta$

(C)  $\frac{3}{2} \int_0^{\pi/2} \cos^2 \theta \, d\theta$

(D)  $3 \int_0^{\pi/2} \cos \theta \, d\theta$

(E)  $3 \int_0^{\pi} \cos \theta \, d\theta$

# Quick Quiz Sections 11.1 – 11.3

You may use a graphing calculator to solve the following problems.

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(B)  $3 \int_0^{\pi} \cos^2 \theta \, d\theta$

(C)  $\frac{3}{2} \int_0^{\pi/2} \cos^2 \theta \, d\theta$

(D)  $3 \int_0^{\pi/2} \cos \theta \, d\theta$

(E)  $3 \int_0^{\pi} \cos \theta \, d\theta$

# Quick Quiz Sections 11.1 – 11.3

2. For what values of  $t$  does the curve given by the parametric equations  $x = t^3 - t^2 - 1$  and  $y = t^4 + 2t^2 - 8t$  have a vertical tangent?

- (A) 0 only
- (B) 1 only
- (C) 0 and  $2/3$  only
- (D) 0,  $2/3$ , and 1
- (E) no value

# Quick Quiz Sections 11.1 – 11.3

2. For what values of  $t$  does the curve given by the parametric equations  $x = t^3 - t^2 - 1$  and  $y = t^4 + 2t^2 - 8t$  have a vertical tangent?

- (A) 0 only
- (B) 1 only
- (C) 0 and  $2/3$  only
- (D) 0,  $2/3$ , and 1
- (E) no value

# Quick Quiz Sections 11.1 – 11.3

3. The length of the path described by the parametric equations  $x = t^2$  and  $y = t$  from  $t = 0$  to  $t = 4$  is given by which integral?

(A)  $\int_0^4 \sqrt{4t+1} dt$

(B)  $2 \int_0^4 \sqrt{t^2+1} dt$

(C)  $\int_0^4 \sqrt{2t^2+1} dt$

(D)  $\int_0^4 \sqrt{4t^2+1} dt$

(E)  $2\pi \int_0^4 \sqrt{4t^2+1} dt$



# Quick Quiz Sections 11.1 – 11.3

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(A)  $\int_0^4 \sqrt{4t+1} dt$

(B)  $2 \int_0^4 \sqrt{t^2 + 1} dt$

(C)  $\int_0^4 \sqrt{2t^2 + 1} dt$

(D)  $\int_0^4 \sqrt{4t^2 + 1} dt$

(E)  $2\pi \int_0^4 \sqrt{4t^2 + 1} dt$

# Chapter Test

1. Given  $x = (1/2) \tan t$ ,  $y = (1/2) \sec t$ ;  $t = \pi / 3$

(a) Find an equation for the line tangent to the curve at the given value of  $t$ .

(b) Find the value of  $\frac{d^2 y}{dx^2}$  at this point.

2. Given  $x = (1/2) \tan t$ ,  $y = (1/2) \sec t$ ,  
find the points at which the tangent to the curve is

(a) horizontal;

(b) vertical.

3. Find equations for the horizontal and vertical lines to the curve  
 $r = 2(1 - \sin \theta)$ ,  $0 \leq \theta \leq 2\pi$ .

# Chapter Test

4. Replace the polar equation  $r\cos\theta = r\sin\theta$  by an equivalent Cartesian equation.
5. Replace the Cartesian equation  $x^2 + y^2 + 5y = 0$  by an equivalent polar equation.
6.  $\mathbf{r}(t) = \langle 4\cos t, \sqrt{2}\sin t \rangle$  is the position vector of a particle moving in the plane at time  $t$ . Find
  - (a) the velocity and acceleration vectors
  - (b) the speed of the particle at  $t = \pi / 4$ .

# Chapter Test Solutions

1. Given  $x = (1/2) \tan t$ ,  $y = (1/2) \sec t$ ;  $t = \pi / 3$

(a) Find an equation for the line tangent to the curve at the given

value of  $t$ . (a)  $y = \frac{\sqrt{3}}{2}x + \frac{1}{4}$

(b) Find the value of  $\frac{d^2 y}{dx^2}$  at this point. (b)  $1/4$

2. Given  $x = (1/2) \tan t$ ,  $y = (1/2) \sec t$ ,

find the points at which the tangent to the curve is

(a) horizontal; (a)  $(0, 1/2)$  and  $(0, -1/2)$

(b) vertical. (b) nowhere

3. Find equations for the horizontal and vertical lines to the curve

$r = 2(1 - \sin \theta)$ ,  $0 \leq \theta \leq 2\pi$ .  $y = 1/2$ ,  $y = -4$ ,  $x = 0$ ,  $x \approx \pm 2.598$

# Chapter Test Solutions

4. Replace the polar equation  $r\cos\theta = r\sin\theta$  by an equivalent Cartesian equation.  $x = y$

5. Replace the Cartesian equation  $x^2 + y^2 + 5y = 0$  by an equivalent polar equation.  $r = -5\sin\theta$

6.  $\mathbf{r}(t) = \langle 4\cos t, \sqrt{2}\sin t \rangle$  is the position vector of a particle moving in the plane at time  $t$ . Find

(a) the velocity and acceleration vectors  $\mathbf{v}(t) = \langle -4\sin t, \sqrt{2}\cos t \rangle$

$\mathbf{a}(t) = \langle -4\cos t, -\sqrt{2}\sin t \rangle$

(b) the speed of the particle at  $t = \pi/4$ . 3